Polytomous Rasch models in SAS

Karl Bang Christensen & Maja Olsbjerg

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The polytomous Rasch model

- Parameter estimation
- Graphics
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   - Parameter estimation
   - Graphics

2. Examples of existing software
Agenda

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3. Implementation in SAS
   - Parameter estimation
   - Graphics. Goodness of fit
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4. Possible extensions
Let $X_1, ..., X_k$ denote the response variables of $k$ items where

$$X_i \in \{0, ..., m_i\}.$$ 

For a latent variable $\theta$ define

$$P(X_i = x_i | \theta) = \frac{\exp(x_i \theta + \eta_{i,x_i})}{\sum_{h=0}^{m_i} \exp(h \theta + \eta_{i,h})}.$$
The Polytomous Rasch model

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Alternative parametrization using thresholds

$$\beta_{i,h} = -(\eta_{i,h} - \eta_{i,h-1}).$$
Assumptions:

- **Unidimensionality:** $\theta \in \mathbb{R}$.

- **Monotonicity:** $\theta \mapsto E(X_i|\theta)$ is increasing for all items $i$.

- **Local independence:**
  
  $$P(X_1 = x_1, \ldots, X_k = x_k|\theta) = \prod_{i=1}^{k} P(X_i = x_i|\theta) \quad \forall \theta \in \mathbb{R}.$$ 

- **No DIF:**
  
  $$P(X_i = x_i|Y, \theta) = P(X_i = x_i|\theta)$$

  for all items $i$ and all exogenous variables $Y$. 

Parameter estimation

- **Item parameters:**
  - Conditional Maximum Likelihood (CML).
  - Pairwise Conditional Maximum Likelihood (PCML).
  - Marginal Maximum Likelihood (MML).

- **Person locations:**
  - Maximum Likelihood Estimation (MLE).
  - Warms Weighted Likelihood Estimation (WLE).
Parameter estimation

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Graphics

- Item characteristic curves
- Item and test information
- Plot of person and item parameters ’Wright map’
- Observed vs. expected mean scores (Goodness of fit)
Examples of existing software

- RUMM (http://www.rummlab.com.au)
- ConQuest (Wu, Adams, Wilson & Haldane)
- DIGRAM (Kreiner)
- R: eRm package (Mair & Hatzinger)
Motivation for implementation in SAS

- SAS is widely used and well-documented.
- Providing software for people without access to proprietary software programs.
- Bringing the methodology to a wider range of researchers.
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SAS macros have already been developed:

- macro %rasch (Christensen & Bjorner)
- macro %AnaQol (Hardouin & Mesbah)
Define vectors of item parameters

$$\eta_i = (\eta_{i,1}, \ldots, \eta_{i,m_i}) \quad i = 1, \ldots, k$$

and response vectors

$$X_v = (X_{v1}, \ldots, X_{vk}) \quad v = 1, \ldots, N.$$ 

Let $g_\omega$ denote the density of $\theta$. The marginal likelihood

$$L(\eta_1, \ldots, \eta_k | x_1, \ldots, x_N) = \prod_{v=1}^{N} \int_{\mathbb{R}} P(X_v = x_v | \theta) g_\omega(\theta) d\theta$$

is maximized numerically.
The NLMIXED (Non-Linear MIXED models) procedure is used for estimation.

- $g_\omega$ density of $\mathcal{N}(0, \sigma^2)$.
- Adaptive Gaussian Quadrature for integral approximation.
- Newton-Raphson algorithm for optimization.

Incomplete item responses not a problem.
MLE of person locations

Sum score $T = \sum_{i=1}^{k} X_i$ sufficient for $\theta$.

$$P(T = t|\theta) = \frac{\exp(\theta t)}{\sum_{s=0}^{m.} \exp(s\theta) \gamma_s} \gamma_t$$

where $m. = \sum_{i=1}^{k} m_i$ and

$$\gamma_t = \gamma_t(\eta_1, \ldots, \eta_k) = \sum_{x. = t} \exp \left( \sum_{i=1}^{k} \sum_{h=0}^{m_i} 1(x_i = h) \eta_i, h \right)$$

where $x. = \sum_{i=1}^{k} x_i$. 
MLE of person locations

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where $x. = \sum_{i=1}^{k} x_i$. Recursion formula:

$$\gamma_t^{(i)} = \frac{\min(m_i, t)}{\sum_{r=t - \sum_{k=1}^{i-1} m_k}^{\min(m_i, t)} \exp(\eta_i, r) \gamma_t^{(i-1)}}$$

facilitates computation.
Solve likelihood equation

\[
T = E(T|\theta) = \sum_t tP(T = t|\theta)
\]

numerically.

- **Complete cases:** likelihood equation solved for each value of \(T\).

- **Incomplete cases:** adapt recursion formula and likelihood equation.
Person and item locations

- Item information for item $i$:
  \[ I_i(\theta) = V(X_i | \theta). \]

- Test information:
  \[ I(\theta) = \sum_{i=1}^{k} I_i(\theta). \]

- Graphical representation 'Wright map'.
Observed mean scores:

- Let $w_\theta$ be the number of persons with $\hat{\theta}_v = \theta$.

- Calculate observed mean scores

\[
\theta \mapsto \frac{1}{w_\theta} \sum_{v: \hat{\theta}_v = \theta} x_{vi}.
\]
Simulated mean scores:

- Sample $\theta_1^{(s)}, \ldots, \theta_N^{(s)}$ from empirical distribution of the estimated person locations.

- Let $w_\theta^{(s)}$ be the number of persons with $\theta_v^{(s)} = \theta$.

- Simulate response vectors $x_{1i}^{(s)}, \ldots, x_{Ni}^{(s)}$ using $P(\cdot | \theta_v^{(s)})$.

- Calculate simulated mean scores

  $\theta \mapsto \frac{1}{w_\theta^{(s)}} \sum_{v: \theta_v^{(s)} = \theta} x_{vi}^{(s)}$.

- Repeat for $s = 1, \ldots, S$. 
Item and latent parameters are estimated by default.

```
%mml(
   DATA=HADS,
   ITEM_NAMES=item_names,
   OUT=MML,
   ICC=NO,
   THRESHOLD=NO,
   INFORMATION=NO,
   WRIGHT=NO,
   FITPLOT=NO
);
```

Pallant, Tennant. British Journal of Clinical Psychology.
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```%mml(
   DATA=HADS,
   ITEM_NAMES=item_names,
   OUT=MML,
   ICC=YES,
   THRESHOLD=NO,
   INFORMATION=NO,
   WRIGHT=NO,
   FITPLOT=NO
);```
Data example

Item characteristic curves: AHADS7
Item and latent parameters are estimated by default.

```
%mml(
DATA=HADS,
ITEM_NAMES=item_names,
OUT=MML,
ICC=YES,
THRESHOLD=YES,
INFORMATION=NO,
WRIGHT=NO,
FITPLOT=NO)
```
Item characteristic curves: AHADS7
Item and latent parameters are estimated by default.

\%
\texttt{mml(} \texttt{DATA=HADS,}
\texttt{ITEM\_NAMES=item\_names,}
\texttt{OUT=MML,}
\texttt{ICC=NO,}
\texttt{THRESHOLD=NO,}
\texttt{INFORMATION=YES,}
\texttt{WRIGHT=NO,}
\texttt{FITPLOT=NO }\texttt{);}
Item characteristic curves and information function

Item AHADS7
Item and latent parameters are estimated by default.

%mml(
   DATA=HADS,
   ITEM_NAME=item_names,
   OUT=MML,
   ICC=NO,
   THRESHOLD=NO,
   INFORMATION=NO,
   WRIGHT=YES,
   FITPLOT=NO
);
Data example

Wright Map

Histogram of person locations (top) and item locations (bottom) on the latent scale
Data example

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%mml(
   DATA=HADS,
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   ICC=NO,
   THRESHOLD=NO,
   INFORMATION=NO,
   WRIGHT=NO,
   FITPLOT=OUI);

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Observed and expected item means:

item AHADS7
Data example

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Possible extensions

- 2-PL model
- Multidimensional model
- Reduced rank parametrization of thresholds
- Latent structures
- Fittests